SIAM J. CONTROL AND OPTIMIZATION
Vol. 25, No. 4, July 1987

001

LINEAR CONTROL THEORY WITH AN $\mathbf{H}_\infty$ OPTIMALITY CRITERION*

BRUCE A. FRANCIS† AND JOHN C. DOYLE‡

Abstract. This expository paper sets out the principal results in $\mathbf{H}_\infty$ control theory in the context of continuous-time linear systems. The focus is on the mathematical theory rather than computational methods.

Key words. $\mathbf{H}_\infty$ control theory, linear systems

AMS(MOS) subject classification. 93

1. Introduction. The subject of this paper is a general regulator problem: a controller is to be designed to regulate the output of a plant subjected to exogenous inputs, such as disturbances, sensor noises and reference signals. A theory for the regulator problem begins by specifying a model of the plant (the model may be a set, to reflect uncertainty), a model of the exogenous inputs, the performance requirements of the controlled system and the allowable class of controllers. For example, two typical regulator theories are the algebraic approach of Pernebo [70] and the Wiener-Hopf approach of Youla, Jabr and Bongiorno [93]. In both these theories the plant is a known time-invariant finite-dimensional linear system and the controller is required to be of this type too. In the algebraic approach the exogenous signal is (after prefiltering) an unknown initial condition, or equivalently a signal of the form $\delta(t)x$, where x is an unknown vector, and the performance requirements are internal stability and asymptotic regulation. In the Wiener-Hopf approach the exogenous signal is (again, after prefiltering) standard white noise, and the performance requirements are internal stability and minimization of the mean-square value of some signal.

In a seminal paper [96], [97], Zames introduced a new theory for the regulator problem. To describe this theory we need a few preliminary mathematical concepts [25], [82]. The Hardy space $\mathbf{H}_\infty$ is the class of matrix-valued functions which are analytic and bounded in the open right half-plane, the $\mathbf{H}_\infty$-norm of such a function, say $F(s)$, being defined as

$$\|F\|_\infty := \sup \sigma_{\max}[F(s)].$$

Here $\sigma_{\max}$ denotes maximum singular value and the supremum is over all s in the open right half-plane, $\operatorname{Re} s > 0$. For such a function the boundary value

$$F(j\omega) := \lim_{\xi \downarrow 0} F(\xi + j\omega)$$

exists for almost all ω and the boundary function is of class $\mathbf{L}_\infty$ (Fatou's theorem). As a consequence of the maximum modulus principle the $\mathbf{H}_\infty$-norm of $F(s)$ equals the $\mathbf{L}_\infty$-norm of the boundary function, i.e.,

$$\|F\|_\infty = \operatorname*{ess\,sup}_{\omega} \sigma_{\max}[F(j\omega)].$$

For example, suppose $F(s)$ is scalar-valued, analytic and bounded in $\operatorname{Re} s > 0$, and continuous on the imaginary axis. Then $\|F\|_\infty$ equals the distance in the complex plane from the origin to the farthest point on the Nyquist plot of F.

* Received by the editors February 2, 1985; accepted for publication (in revised form) April 3, 1986.
† Department of Electrical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4.
‡ Department of Electrical Engineering, California Institute of Technology, Pasadena, California 91125.

A fundamental fact is that the $\mathbf{L}_2[0,\infty)$-gain of a causal time-invariant linear system equals the $\mathbf{H}_\infty$-norm of its transfer function. To state this more precisely introduce the space $\mathbf{L}_2[0,\infty)$ of vector-valued square-integrable functions. The norm on $\mathbf{L}_2[0,\infty)$ is

$$\|x\|_2 := \left[\int_0^\infty x(t)^* x(t)\, dt\right]^{1/2},$$

where * denotes complex-conjugate transpose. The Laplace transform of $x(t)$ in $\mathbf{L}_2[0,\infty)$, denoted with abuse of notation by $x(s)$, belongs to the Hardy space $\mathbf{H}_2$ of functions analytic in $\operatorname{Re} s > 0$ and satisfying the condition

$$\sup_{\xi>0} \int_{-\infty}^{\infty} x(\xi + j\omega)^* x(\xi + j\omega)\, d\omega < \infty.$$

Such functions also have boundary values almost everywhere and the $\mathbf{H}_2$-norm is

$$\|x\|_2 := \left[(2\pi)^{-1} \int_{-\infty}^{\infty} x(j\omega)^* x(j\omega)\, d\omega\right]^{1/2}.$$

The Laplace transform is a Hilbert space isomorphism from $\mathbf{L}_2[0,\infty)$ onto $\mathbf{H}_2$ (the Paley–Wiener theorem). Now consider a causal time-invariant linear system having a transfer matrix $F(s)$. Suppose $F \in \mathbf{H}_\infty$. Then $Fx \in \mathbf{H}_2$ whenever $x \in \mathbf{H}_2$, and moreover,

$$\|F\|_\infty = \sup\{\|Fx\|_2 : x \in \mathbf{H}_2, \|x\|_2 = 1\}. \tag{1}$$

The $\mathbf{H}_\infty$-norm arises in the regulator problem primarily under two circumstances: when there are sets of exogenous signals and when there is plant uncertainty.

Consider first an example of a tracking problem in which a plant output is to track a reference signal. Suppose, for simplicity, that these two signals are scalar-valued, and let $F(s)$ denote the transfer function from the reference input to the tracking error (reference minus output). Assume the system is stable in the sense that $F \in \mathbf{H}_\infty$. Control designs are often based on test inputs, sinusoids being the natural ones in the frequency domain. Suppose the reference signal is allowed to be any sinusoid of amplitude no greater than 1 and of frequency belonging to some interval Ω. An appropriate performance measure might then be

$$\operatorname*{ess\,sup}_{\omega\in\Omega} |F(j\omega)|,$$

this equaling the maximum amplitude of the tracking error. Let $W(s)$ be an $\mathbf{H}_\infty$-function such that

$$|W(j\omega)| = 1, \quad \omega \in \Omega, \qquad |W(j\omega)| = \varepsilon, \quad \omega \notin \Omega.$$

For small ε the performance measure is approximated by the $\mathbf{H}_\infty$-norm $\|WF\|_\infty$. (For a nontrivial function to be analytic in the right half-plane, its magnitude cannot be zero on a subset of the imaginary axis of positive measure (F. and M. Riesz' theorem); hence the necessity of introducing ε.)

The previous example shows how an $\mathbf{H}_\infty$-norm performance measure can arise from consideration of a set of exogenous inputs, namely sinusoids. Another way to arrive at the same performance measure is with inputs belonging to $\mathbf{L}_2[0,\infty)$. Continuing with scalar-valued signals, suppose the reference input is allowed to be any function in the class

$$\{x : x = Wv \quad \text{for some } v \in \mathbf{H}_2, \|v\|_2 \leq 1\},$$

where $W, W^{-1} \in \mathbf{H}_\infty$; that is, the reference signal class consists of all x in $\mathbf{H}_2$ such that

$$(2\pi)^{-1} \int_{-\infty}^{\infty} |x(j\omega)|^2 |W(j\omega)|^{-2}\, d\omega \leq 1. \tag{2}$$

This inequality can be interpreted as a constraint on the weighted energy of x: the energy-density spectrum $|x(j\omega)|^2$ is weighted by the factor $|W(j\omega)|^{-2}$. For example, if $|W(j\omega)|$ were relatively large on a certain frequency band and relatively small off it, then (2) would generate a class of signals having their energy concentrated on that band. This could be useful in representing, for example, a class of narrowband signals whose spectra are confined to a common frequency band. If $F(s)$ again denotes the transfer function from x to the tracking error, then by virtue of (1), $\|WF\|_\infty$ equals the maximum $\mathbf{H}_2$-norm of the tracking error (i.e., the square root of its energy).

The problem of robust stabilization can also lead to an $\mathbf{H}_\infty$ criterion. In this introductory section we consider a simplified version of the problem; a fuller account is given in § 2. The block diagram in Fig. 1(a) shows a plant and a controller with transfer matrices $P(s)+\Delta P(s)$ and $K(s)$ respectively; P represents the nominal plant and ΔP an unknown perturbation, usually caused by unmodeled dynamics or parameter variations. Suppose, for simplicity, that P, ΔP, and K are rational, P and ΔP are strictly proper, K is proper, and P and ΔP are analytic in $\operatorname{Re} s \geqq 0$. Suppose also that the system is internally stable for $\Delta P = 0$. How large can ΔP be so that internal stability is maintained?

One method which is used to obtain a transfer function model of a physical system is a frequency response experiment. This yields gain and phase estimates at several frequencies, which in turn provide an upper bound for the norm of $\Delta P(j\omega)$ at several values of ω. Suppose r is a scalar-valued $\mathbf{H}_\infty$-function such that

$$\sigma_{\max}[\Delta P(j\omega)] < |r(j\omega)| \quad \text{for all } \omega,$$

or equivalently

$$\|r^{-1}\Delta P\|_\infty < 1. \tag{3}$$

How large can r be so that internal stability is maintained?

Simple loop transformations lead from Fig. 1(a) to Fig. 1(b) to Fig. 1(c). Since the nominal feedback system is internally stable, $K(I-PK)^{-1} \in \mathbf{H}_\infty$. The small gain theorem [78], [95] says that the system in Fig. 1(c) will be internally stable provided the loop gain is less than unity, i.e.,

$$\|\Delta PK(I-PK)^{-1}\|_\infty < 1. \tag{4}$$

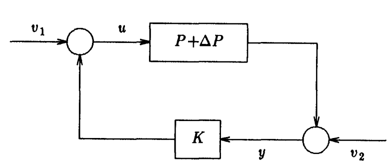

FIG. 1(a). *Feedback system with perturbed plant.*

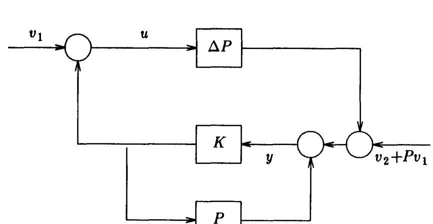

FIG. 1(b). *Loop transformation.*

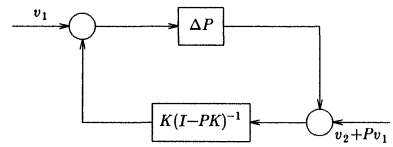

FIG. 1(c). *Loop transformation.*

In view of (3) a sufficient condition for (4) is

(5) $$\|rK(I-PK)^{-1}\|_\infty \leqq 1.$$

We conclude that an $\mathbf{H}_\infty$-norm bounded on a weighted closed-loop transfer matrix is sufficient for robust stability. (Condition (5) is actually necessary for internal stability for all perturbations satisfying (3) ([10], [19]).)

The problem treated in this paper concerns the system in Fig. 2. The signals w, u, z and y are vector-valued and denote, respectively, the exogenous signal (disturbances, sensor noises, reference inputs, etc.), the control signal, the signal to be regulated (tracking errors, plant outputs to be attenuated, weighted actuator outputs, etc.) and the measured signal. The transfer matrices G and K represent the plant and controller respectively. It is assumed that G is real-rational, proper and given; a real-rational proper K is sought to minimize the $\mathbf{H}_\infty$-norm of the transfer matrix from w to z under the constraint of internal stability.

For ease of reference let us call the problem just stated the *standard* ($\mathbf{H}_\infty$) *problem.* It must be emphasized that a controller is designed for a given nominal G; uncertainty in G is not a consideration. (However, it may already be evident, and will be shown in § 2, that the robust stabilization problem can be recast as a standard problem.) There now exists a reasonably complete solution to the standard problem. The purpose of this paper is to set out the principal results in the context of continuous-time linear systems. The focus is on the mathematical theory rather than computational methods. For the latter the reader may consult [21], [50].

Inclusion of plant uncertainty into the $\mathbf{H}_\infty$ problem increases its difficulty considerably. Let us suppose that uncertainty is introduced in the following general way: G can be any element in a family $\mathbf{G}$. We could then try to find a controller to minimize the maximum $\mathbf{H}_\infty$-norm of the transfer matrix from w to z, the maximum taken over all G in $\mathbf{G}$. For this problem Zames [97] has obtained qualitative results for a simple feedback configuration, showing how performance degrades as uncertainty increases, and Doyle [20] has introduced the concept of structured uncertainty, where the elements of $\mathbf{G}$ have specified structures as well as norm constraints; the $\mathbf{H}_\infty$ problem with structured uncertainty can be reduced to a family of standard problems, thus providing further motivation for the latter.

This introductory section concludes with a brief survey of the literature. The first papers on the subject of $\mathbf{H}_\infty$-norm optimization of systems are those of Helton [47],

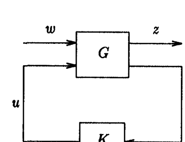

FIG. 2. *The standard configuration.*

Tannenbaum [83] and Zames [96]. The important papers of Sarason [79] and Adamjan, Arov and Krein [1] established connections between operator theory and complex function theory, in particular, $\mathbf{H}_\infty$-functions; Helton [47] showed that these two mathematical subjects have useful applications in electrical engineering, namely, in broadband matching. Tannenbaum [83], [84] used (Nevanlinna–Pick) interpolation theory to attack the problem of stabilizing a plant with an unknown gain. And Zames [96] formulated the problem of sensitivity reduction by feedback as an optimization problem with an operator norm, in particular, an $\mathbf{H}_\infty$-norm. The latter paper was amplified in the seminal paper [97].

The fact that the $\mathbf{L}_2$-gain of a system equals the $\mathbf{H}_\infty$-norm of its transfer function (i.e. (1)) was used by Zames [94] in his pioneering work on nonlinear system theory. This fact is also central in $\mathbf{L}_2$-stability theory, such as the circle criterion (see e.g. [17]).

Motivation for the $\mathbf{H}_\infty$ approach with regard to modeling the exogenous signals is discussed in [20], [21], [24], [90], [97], [98] and with regard to plant uncertainty in [21], [62], [73], [97]. Classical frequency-domain performance specifications also lead to an $\mathbf{H}_\infty$ criterion as shown in [48], [75]. The robust stabilization problem discussed above is treated in [22], [24], [44], [54], [58], [73], [84], [85], [88]–[90]. Various versions of the standard problem for finite-dimensional time-invariant systems are covered in [35], [41], [56], [62], [86], [90], [98] in the single-input/single-output case and in [8], [9], [12], [13], [20]–[24], [32]–[34], [36], [37], [48], [49], [60], [61], [71], [74], [77], [87], [90]–[92], [97], [99] in the multivariable case. The mathematical tools primarily used in these references are Nevanlinna–Pick interpolation theory [16], the operator theory of Sarason [79] and Adamjan, Arov and Krein [1], [2] and the geometric theory of Ball and Helton [4]. The standard problem is extended beyond the finite-dimensional time-invariant case in [14], [15], [26]–[31], [42], [54], [55], [57]. References [7], [23], [35], [38]–[40], [51], [66]–[69] present performance bounds for systems designed according to an $\mathbf{H}_\infty$ criterion. Algorithms for computing optimal controllers are contained in [21], [50], [76], [80]. Part of the computation in [21] involves solving a special model-reduction problem, for which state-space algorithms are presented in [5], [6], [43], [59], [81]. Finally, the $\mathbf{H}_\infty$ approach is compared with the Wiener–Hopf approach in [21], [45], [97], [98].

2. The standard problem. The standard problem pertains to Fig. 2. It is *assumed* that G is real-rational and proper (analytic at $s=\infty$). Partition it as

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

so that the equations corresponding to Fig. 2 are

$$z=G_{11}w+G_{12}u, \qquad y=G_{21}w+G_{22}u, \qquad u=Ky.$$

Now eliminate u and y to get that the transfer matrix from w to z is a linear fractional transformation of K:

$$z=[G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21}]w.$$

It simplifies the theory to guarantee that the rational matrix $I-G_{22}K$ is invertible for every proper real-rational K. A simple sufficient condition for this is that G_{22} be strictly proper (equal to zero at $s=\infty$). Accordingly, this will be *assumed* hereafter.

To define what it means for K to stabilize G, introduce two fictitious inputs v_1 and v_2 as in Fig. 3. It is easy to show that the nine transfer matrices from the three

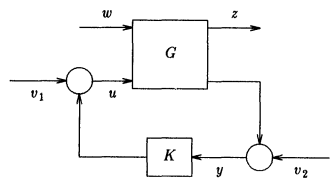

FIG. 3. *System for definition of stability.*

inputs w, v_1, v_2 to the three signals z, u, y exist and are proper; if they belong to $\mathbf{H}_\infty$, then K *stabilizes* G. This is the usual notion of internal stability. An equivalent definition in terms of state-space models is as follows. Take minimal state-space realizations of G and K and in Fig. 2 set the input w to zero. Then K stabilizes G if and only if the state vectors of G and K tend to zero from every initial condition.

The *standard problem* is this: find a real-rational proper K to minimize the $\mathbf{H}_\infty$-norm of the transfer matrix from w to z under the constraint that K stabilize G.

Following are three examples of the standard problem.

2.1. A model-matching problem. In Fig. 4 the transfer matrix T_1 represents a "model" which is to be matched by the cascade T_2QT_3 of three transfer matrices T_2, T_3 and Q. Here, T_i $(i=1-3)$ are given and the "controller" Q is to be designed. Let $\mathbf{RH}_\infty$ denote the space of real-rational matrices in $\mathbf{H}_\infty$, that is, the space of real-rational proper matrices which are stable, i.e., analytic in $\operatorname{Re} s \geqq 0$. It is assumed that $T_i \in \mathbf{RH}_\infty$ $(i=1-3)$ and it is required that $Q \in \mathbf{RH}_\infty$. Thus the four blocks in Fig. 4 represent stable linear systems.

For our purposes the *model-matching criterion* is

$$\sup\{\|z\|_2 \colon w \in \mathbf{H}_2, \|w\|_2 \leqq 1\} = \text{minimum}.$$

Thus the energy of the error z is to be minimized for the worst input w of unit energy. In view of (1) an equivalent criterion is

$$\|T_1 - T_2QT_3\|_\infty = \text{minimum}.$$

This model-matching problem can be cast as a standard problem by defining

$$G := \begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix}, \qquad K := -Q,$$

so that Fig. 4 becomes equivalent to Fig. 2. The constraint that K stabilize G is then equivalent to the constraint that $Q \in \mathbf{RH}_\infty$.

This version of the model-matching problem is not very important per se; its significance in the context of this paper arises from the fact that the standard problem can be transformed to the model-matching problem (§ 3), which is simpler.

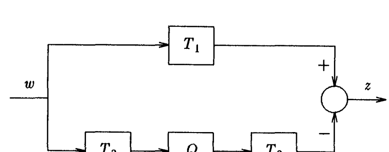

FIG. 4. *Model-matching.*

2.2. A tracking problem [90], [91]. Figure 5 shows a plant P whose output, v, is to track a reference signal r. The plant input, u, is generated by passing r and v through controllers C_1 and C_2 respectively. It is postulated that r is not a known fixed signal, but, as in the introduction, may be modeled as belonging to the class

$$\{r: r = Ww \text{ for some } w \in \mathbf{H}_2, \|w\|_2 \leqq 1\}.$$

Here P and W are given and C_1 and C_2 are to be designed. These four transfer matrices are assumed to be real-rational and proper.

The tracking error signal is $r - v$. Let us take the cost function to be

$$(\|r-v\|_2^2 + \|\rho u\|_2^2)^{1/2}, \tag{6}$$

where ρ is a positive scalar weighting factor. The reason for including ρu in (6) is to ensure the existence of an optimal proper controller; for $\rho = 0$ "optimal" controllers tend to be improper. Note that (6) equals the $\mathbf{H}_2$-norm of

$$z := \begin{bmatrix} r-v \\ \rho u \end{bmatrix}.$$

Thus the *tracking criterion* is taken to be

$$\sup\{\|z\|_2 \colon w \in \mathbf{H}_2, \|w\|_2 \leqq 1\} = \text{minimum}.$$

The equivalent standard problem is obtained by defining

$$y := \begin{bmatrix} r \\ v \end{bmatrix}, \qquad K := [C_1 \quad C_2],$$

$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \tag{7a}$$

$$G_{11} := \begin{bmatrix} W \\ 0 \end{bmatrix}, \qquad G_{12} := \begin{bmatrix} -P \\ \rho I \end{bmatrix}, \tag{7b}$$

$$G_{21} := \begin{bmatrix} W \\ 0 \end{bmatrix}, \qquad G_{22} := \begin{bmatrix} 0 \\ P \end{bmatrix}. \tag{7c}$$

2.3. A robust stabilization problem [58], [73], [88]. This example has already been discussed in the Introduction. The system under consideration is shown in Fig. 1(a). Assume P is a strictly proper nominal plant and let r be a scalar-valued (radius) function in $\mathbf{RH}_\infty$. Now define a family $\mathbf{P}$ of neighboring plants to be the set of all strictly proper real-rational matrices $P + \Delta P$ having the same number (in terms of McMillan degree) of poles in Re $s \geqq 0$ as P has, where the perturbation ΔP satisfies the bound

$$\sigma_{\max}[\Delta P(j\omega)] < |r(j\omega)| \quad \text{for all } \omega.$$

For a real-rational proper K the *robust stability criterion* is that K stabilize all plants in $\mathbf{P}$. Stability means internal stability, that the four transfer matrices in Fig. 1(a) from v_1, v_2 to u, y all belong to $\mathbf{RH}_\infty$.

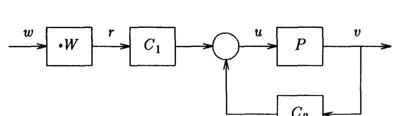

FIG. 5. *Tracking.*

We saw in the Introduction that robust stability is guaranteed by a small gain condition.

LEMMA 2.1 [10], [19]. *A real-rational proper K stabilizes all plants in* $\mathbf{P}$ *if and only if K stabilizes the nominal plant P and*

$$\|rK(I-PK)^{-1}\|_\infty \leqq 1.$$

We can convert to the set-up of the standard problem by defining G so that in Fig. 2 the transfer matrix from w to z equals $rK(I-PK)^{-1}$. This is accomplished by

$$G := \begin{bmatrix} 0 & rI \\ I & P \end{bmatrix}.$$

Then Lemma 2.1 implies that the following two conditions are equivalent: K achieves robust stability for the original system (Fig. 1(a)); in Fig. 2 K stabilizes G and puts the transfer matrix from w to z inside the closed unit ball of $\mathbf{H}_\infty$.

There are several other examples of the standard problem and several other problems which are equivalent to the standard problem [21].

3. From the standard problem to a model-matching problem. In this section it is shown that the standard problem can be reduced to the model-matching problem of the first example. The procedure is to parametrize, via a parameter matrix Q in $\mathbf{RH}_\infty$, all K's which stabilize G. Then Fig. 2 can be transformed into Fig. 4, where the T_i's depend only on G. The parametrization employed in this section is due to Youla et al. [93] as modified by Desoer et al. [18]. Previous work on stability theory for the system in Fig. 2 was carried out by Pernebo [70], Cheng and Pearson [11], and Antoulas [3]; Nett [65] treated a more general setup (K has an additional input and output).

When K stabilizes G can be characterized in terms of coprime factorizations over the ring $\mathbf{RH}_\infty$ of stable proper real-rational functions. Factor G and K as

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \qquad K = UV^{-1} = \tilde{V}^{-1}\tilde{U}.$$

The matrices N and M belong to $\mathbf{RH}_\infty$ and are *right-coprime.* This means that, if X is a square matrix in $\mathbf{RH}_\infty$ which is a right divisor of both N and M, i.e.,

$$N = YX, \qquad M = ZX \quad \text{for some } Y, Z \quad \text{in } \mathbf{RH}_\infty,$$

then X is a unit of $\mathbf{RH}_\infty$, i.e. $X^{-1} \in \mathbf{RH}_\infty$. Such N and M constitute a *right-coprime factorization* (rcf) of G. Analogously, $\tilde{N}$ and $\tilde{M}$ are left-coprime and constitute a left-coprime factorization (lcf) of G. Similar remarks apply to the factorization of K. Such factorizations are known to exist [90].

PROPOSITION 3.1. *The following are equivalent statements about the proper transfer matrix K:*

(i) *K stabilizes G,*

(ii) $\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0 \;\; I]N & V \end{bmatrix}$ *is a unit of* $\mathbf{RH}_\infty$,

(iii) $\begin{bmatrix} \tilde{M} & \tilde{N}\begin{bmatrix} 0 \\ I \end{bmatrix} \\ \tilde{U}[0 \;\; I] & \tilde{V} \end{bmatrix}$ *is a unit of* $\mathbf{RH}_\infty$.

The idea underlying the equivalence of (i) and (ii) is simply that the determinant of the matrix in (ii) is the least common denominator (in $\mathbf{RH}_\infty$) of all the transfer functions from w, v_1, v_2 to z, u, y; hence the determinant must be a unit for all these transfer functions to belong to $\mathbf{RH}_\infty$, and conversely.

The transfer matrix G is *stabilizable* if there exists a K which stabilizes it. Not every G is stabilizable; an obvious nonstabilizable G is $G_{12}=0$, $G_{21}=0$, $G_{22}=0$, G_{11} unstable. Proposition 3.1 provides a test for stabilizability. For example, in terms of N and M, G is stabilizable if and only if suitable U and V exist satisfying condition (ii). Such a consideration readily leads to the following.

PROPOSITION 3.2. *The following are equivalent*:

(i) *G is stabilizable*;

(ii) *M, $[0\ \ I]N$ are right-coprime and M, $\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]$ are left-coprime*;

(iii) *$\tilde{M}$, $\tilde{N}\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]$ are left-coprime and M, $[0\ \ I]$ are right-coprime.*

In terms of a state-space model G is stabilizable if and only if, roughly speaking, its unstable modes are controllable from u and observable from y (Fig. 2). For example, right-coprimeness of M, $[0\ \ I]N$ can be interpreted as a frequency-domain stabilizability condition.

Hereafter, G will be *assumed* to be stabilizable. To recap, G is assumed so far to be real-rational, proper with G_{22} strictly proper, and stabilizable. Intuitively, stabilizability of G implies that G and G_{22} share the same unstable modes, so that to stabilize G it is enough to stabilize G_{22}. The controller K *stabilizes* G_{22} if, in Fig. 6, the four transfer matrices from v_1, v_2 to u, y are stable.

PROPOSITION 3.3. *K stabilizes G if and only if K stabilizes G_{22}.*

The next step is to parametrize all K's stabilizing G_{22}. For this it is convenient to introduce a special rcf and lcf of G_{22}:

$$G_{22}=N_2M_2^{-1}=\tilde{M}_2^{-1}\tilde{N}, \tag{8}$$

$$\begin{bmatrix}\tilde{X}_2 & -\tilde{Y}_2\\ -\tilde{N}_2 & \tilde{M}_2\end{bmatrix}\begin{bmatrix}M_2 & Y_2\\ N_2 & X_2\end{bmatrix}=I. \tag{9}$$

The eight matrices introduced in (8) and (9) all belong to $\mathbf{RH}_\infty$; their existence is proved in § 4. Equation (9) is known as a generalized Bezout identity; its satisfaction guarantees that N_2, M_2 are right-coprime and $\tilde{N}_2$, $\tilde{M}_2$ are left-coprime. Equations (8) and (9) constitute a generalization of the usual polynomial matrix-fraction description [18], [53], [90].

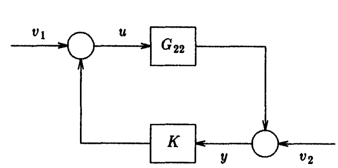

FIG. 6. *Part of standard configuration.*

THEOREM 3.1 [18], [90], [93]. *The following formulas parametrize all proper K's which stabilize G_{22}*:

$$K=(Y_2-M_2Q)(X_2-N_2Q)^{-1} \tag{10}$$

$$=(\tilde{X}_2-Q\tilde{N}_2)^{-1}(\tilde{Y}_2-Q\tilde{M}_2),\qquad Q\in\mathbf{RH}_\infty. \tag{11}$$

The right-hand sides of (10) and (11) constitute an rcf and lcf of K respectively; the inverses exist for every Q in $\mathbf{RH}_\infty$ (because G_{22} is strictly proper). As Q varies over all stable proper matrices, the formulas generate all possible stabilizing K's.

The final step is to determine the transfer matrix from w to z in Fig. 2 when K is given by formulas (10) and (11). Define

$$T_1 := G_{11} + G_{12} Y_2 \tilde{M}_2 G_{21}, \tag{12a}$$

$$T_2 := G_{12} M_2, \tag{12b}$$

$$T_3 := \tilde{M}_2 G_{21}. \tag{12c}$$

It can be proved that $T_i \in \mathbf{RH}_\infty$ $(i = 1-3)$.

THEOREM 3.2 [21]. *With K as in* (10), (11), *the transfer matrix from w to z equals* $T_1 - T_2 Q T_3$.

We conclude from this theorem that *the standard problem reduces to the model-matching problem* of finding matrices Q in $\mathbf{RH}_\infty$ to minimize $\|T_1 - T_2 Q T_3\|_\infty$. A solution Q to the model-matching problem yields a solution K to the standard problem via formulas (10) and (11).

A special case is when G is itself stable. In (8) and (9) we may then take

$$N_2 = \tilde{N}_2 = G_{22},$$

$$M_2, \tilde{M}_2, X_2, \tilde{X}_2 \text{ all} = I,$$

$$Y_2 = 0, \qquad \tilde{Y}_2 = 0,$$

in which case (10) and (11) become [97]

$$\begin{aligned} K &= -Q(I - G_{22}Q)^{-1} \\ &= -(I - QG_{22})^{-1} Q \end{aligned}$$

and (12) produces

$$T_1 = G_{11}, \qquad T_2 = G_{12}, \qquad T_3 = G_{21}.$$

4. State-space computations. The reduction in the previous section was developed using transfer matrix models. However, as a pratical matter the computations are quite easily and reliably performed using state-space models. This section describes how to obtain state-space realizations of the matrices T_i $(i = 1-3)$ starting from a state-space realization of G. The formulas are very simple and they provide a fundamental link (Theorem 4.1) between the stability result of Theorem 3.1 and observer-based stability theory. Some of the results of this section are contained in [64].

We begin with a minimal realization of $G(s)$,

$$G(s) = D + C(s - A)^{-1} B, \qquad A, B, C, D \text{ real matrices.}$$

It is convenient to introduce a new data structure:

$$[A, B, C, D] := D + C(s - A)^{-1} B.$$

Since the input and output of G are partitioned as

$$\begin{bmatrix} w \\ u \end{bmatrix} \text{ and } \begin{bmatrix} z \\ y \end{bmatrix},$$

the matrices B, C, and D have corresponding partitions,

$$B=[B_1 \quad B_2], \qquad C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \qquad D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Thus

$$G_{ij}(s)=[A, B_j, C_i, D_{ij}], \qquad i, j=1, 2.$$

Note that $D_{22}=0$ because G_{22} is strictly proper. It follows from the stabilizability of G that the pair (A, B_2) is stabilizable and the pair (C_2, A) is detectable.

The objective now is to specify state-space realizations of transfer matrices N_2, M_2, etc. belonging to $\mathbf{RH}_\infty$ and satisfying (8) and (9). Denote the state, input and output vectors of G_{22} by x, u and y respectively, so that $y=G_{22}u$ and

$$\dot{x}=Ax+B_2u, \tag{13a}$$

$$y=C_2x. \tag{13b}$$

Next, choose a real matrix F so that $A_F := A+B_2F$ is stable (all eigenvalues in Re $s<0$) and define the vector $v := u-Fx$. Then from (13) we get

$$\dot{x}=A_Fx+B_2v, \qquad u=Fx+v, \qquad y=C_2x.$$

The transfer matrix from v to u is

$$M_2(s) := [A_F, B_2, F, I] \tag{14}$$

and that from v to y is

$$N_2(s) := [A_F, B_2, C_2, 0]. \tag{15}$$

Therefore $G_{22}=N_2M_2^{-1}$. Similarly, by choosing a real matrix H so that $A_H := A+HC_2$ is stable and defining

$$\tilde{M}_2(s) := [A_H, H, C_2, I], \qquad \tilde{N}_2(s) := [A_H, B_2, C_2, 0],$$

we obtain $G_{22}=\tilde{M}_2^{-1}\tilde{N}_2$. Thus (8) is satisfied.

Now introduce an observer-based controller, denoted by $K_o(s)$, to stabilize G_{22}. The familiar equations for K_o are

$$\dot{\hat{x}}=A\hat{x}+B_2u+H(C_2\hat{x}-y), \qquad u=F\hat{x},$$

or equivalently,

$$\dot{\hat{x}}=(A+B_2F+HC_2)\hat{x}-Hy, \qquad u=F\hat{x}.$$

Thus

$$K_o(s)=[A+B_2F+HC_2, -H, F, 0].$$

Obtaining factorizations of K_o analogous to those just obtained for G_{22}, we arrive at the equations

$$K_o=Y_2X_2^{-1}=\tilde{X}_2^{-1}\tilde{Y}_2, \tag{16}$$

where

$$X_2(s) := [A_F, -H, C_2, I], \tag{17}$$

$$Y_2(s) := [A_F, -H, F, 0], \tag{18}$$

$$\tilde{X}_2(s) := [A_H, -B_2, F, I], \qquad \tilde{Y}_2(s) := [A_H, -H, F, 0].$$

Routine algebra verifies (9).

Formula (16) provides just one controller which stabilizes G_{22}, whereas formulas (10) and (11) in Theorem 3.1, having the additional stable matrix Q, generate all stabilizing controllers. These two results can be used to show that every stabilization procedure amounts to adding stable dynamics to the plant and then using an observer-based controller to stabilize the result. The precise statement is as follows.

THEOREM 4.1 [21]. *Suppose K stabilizes*

$$G_{22}(s)=[A, B_2, C_2, 0].$$

Then G_{22} can be embedded in a system $[A_e, B_e, C_e, 0]$, where

$$A_e:=\begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix}, \qquad B_e:=\begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \qquad C_e:=[C_2 \ \ 0]$$

and A_a is stable, such that K has the form

$$K(s)=[A_e+B_eF_e+H_eC_e, -H_e, F_e, 0],$$

where $A_e+B_eF_e$ and $A_e+H_eC_e$ are stable.

Now consider the transfer matrices T_i defined in (12). We have obtained realizations of all the matrices on the right-hand sides of (12). Algebraic manipulations lead to the realizations

(19a)
$$T_1(s)=[\underset{\sim}{A}_1, \underset{\sim}{B}_1, \underset{\sim}{C}_1, D_{11}],$$

$$\underset{\sim}{A}_1=\begin{bmatrix} A_F & -B_2F \\ 0 & A_H \end{bmatrix}, \qquad \underset{\sim}{B}_1=\begin{bmatrix} B_1 \\ B_1+HD_{21} \end{bmatrix},$$

$$\underset{\sim}{C}_1=[C_1+D_{12}F \quad -D_{12}F],$$

(19b)
$$T_2(s)=[A_F, B_2, C_1+D_{12}F, D_{12}],$$

(19c)
$$T_3(s)=[A_H, B_1+HD_{21}, C_2, D_{21}].$$

5. Model-matching theory. This section treats the model-matching problem finding matrices Q in $\mathbf{RH}_\infty$ to minimize the model-matching error $\|T_1-T_2QT_3\|_\infty$ where $T_i \in \mathbf{RH}_\infty$ $(i=1-3)$.

5.1. Existence of a solution. Define the *infimal model-matching error*

(20)
$$\alpha:=\inf\{\|T_1-T_2QT_3\|_\infty: Q\in \mathbf{RH}_\infty\}.$$

The natural question to answer first is when is this infimum achieved. The following provides a mild sufficient condition.

THEOREM 5.1 (e.g. [33]). *The infimum in* (20) *is achieved if the ranks of the two matrices $T_2(j\omega)$ and $T_3(j\omega)$ are constant for all* $0\leqq\omega\leqq\infty$.

These rank conditions will be *assumed* to hold for the remainder of § 5. (In applications they do hold for well-defined problems.) In general there is a family of optimal Q's, i.e., Q's achieving the infimum. Moreover, the model-matching error cannot be reduced by a nonrational Q, i.e.,

$$\min\{\|T_1-T_2QT_3\|_\infty: Q\in \mathbf{RH}_\infty\}=\min\{\|T_1-T_2QT_3\|_\infty: Q\in \mathbf{H}_\infty\}.$$

The latter minimum equals the distance in $\mathbf{H}_\infty$ from T_1 to the subspace

$$T_2\mathbf{H}_\infty T_3:=\{T_2QT_3: Q\in \mathbf{H}_\infty\}.$$

Hence

(21)
$$\alpha=\operatorname{dist}(T_1, T_2\mathbf{H}_\infty T_3).$$

The rank conditions in Theorem 5.1 suffice to make $T_2\mathbf{H}_\infty T_3$ closed in the weak-star topology of $\mathbf{H}_\infty$, and this in turn guarantees the existence of a matrix in $T_2\mathbf{H}_\infty T_3$ closest to an arbitrary T_1.

The model-matching problem is relatively easy when the T_i's are scalar-valued. The main results can be summarized as follows. A function $f(s)$ in $\mathbf{RH}_\infty$ is said to be an *inner function* if $f(s)f(-s)=1$. The zeros of such a function all lie in $\operatorname{Re} s>0$; the number of its zeros will be called its *degree.*

PROPOSITION 5.1 [98] (Scalar-valued case). *The infimum in* (20) *is achieved if T_2T_3 has no zeros on the extended imaginary axis. In this case the optimal Q is unique and is uniquely determined by the following property: $T_1-T_2QT_3$ is a scalar multiple of an inner function of degree less than the number of zeros of T_2T_3 in* $\operatorname{Re} s>0$.

5.2. A formula for the minimal model-matching error. This subsection shows that α equals the norm of a certain operator, a Hankel operator in a special case. First, it will be shown that α can be expressed in the form

$$\alpha=\operatorname{dist}\left(R, \begin{bmatrix} I \\ 0 \end{bmatrix}\mathbf{H}_\infty[I\ \ 0]\right)$$

where the matrix R belongs to $\mathbf{RL}_\infty$ (the space of real-rational $\mathbf{L}_\infty$-matrices) and depends only on the T_i's. For this maneuver we need the concepts of inner and outer matrices [82].

Introduce the notation $F^\sim(s):=F(-s)'$ for a matrix-valued function $F(s)$, where prime denotes transpose. A matrix F in $\mathbf{RH}_\infty$ is an *inner matrix* if $F^\sim F=I$ and an *outer matrix* if it has full row rank in $\operatorname{Re} s>0$; it is termed *co-inner* or *co-outer* if its transpose is inner or outer, respectively.

LEMMA 5.1. *For each matrix F in $\mathbf{RH}_\infty$ there exist inner, outer, co-inner and co-outer matrices F_i, F_o, F_{ci}, F_{co} respectively, such that*

$$F=F_iF_o=F_{co}F_{ci}.$$

If F has constant rank on the extended imaginary axis, then F_o has a right-inverse in $\mathbf{RH}_\infty$ and F_{co} has a left-inverse in $\mathbf{RH}_\infty$.

Returning to the model-matching problem, introduce inner, outer, co-inner and co-outer matrices as follows:

$$T_2=(T_2)_i(T_2)_o, \qquad T_3=(T_3)_{co}(T_3)_{ci}.$$

(In going from the standard problem to the model-matching problem, it is possible to arrange that T_2 is automatically inner and T_3 is automatically co-inner [21].) Lemma 5.1, together with the assumptions on T_2 and T_3, implies that $(T_2)_o$ is right-invertible over $\mathbf{RH}_\infty$ and $(T_3)_{co}$ is left-invertible over $\mathbf{RH}_\infty$. Thus the mapping

$$Q\rightarrow (T_2)_oQ(T_3)_{co}$$

on $\mathbf{H}_\infty$ is surjective. Hence from (21) we get the expression

$$\alpha=\operatorname{dist}(T_1,(T_2)_i\mathbf{H}_\infty(T_3)_{ci}).$$

To simplify notation define

$$U_i:=(T_2)_i, \qquad U_{ci}:=(T_3)_{ci}$$

so that

$$\alpha=\operatorname{dist}(T_1, U_i\mathbf{H}_\infty U_{ci}). \tag{22}$$

Being inner, U_i has the property that $U_i^\sim U_i = I$. This in turn implies that

$$E^\sim E = I, \tag{23}$$

where

$$E := \begin{bmatrix} U_i^\sim \\ I - U_i U_i^\sim \end{bmatrix}.$$

It is a consequence of (23) that the norm of an $\mathbf{L}_\infty$-matrix is unchanged if the matrix is pre-multiplied by E. Similarly, $L^\sim L = I$, where

$$L := \begin{bmatrix} U_{ci} \\ I - U_{ci}^\sim U_{ci} \end{bmatrix},$$

and post-multiplication by $L^\sim$ preserves the $\mathbf{L}_\infty$-norm. Hence (22) yields

$$\alpha = \text{dist}\,(ET_1L^\sim, EU_i\mathbf{H}_\infty U_{ci}L^\sim).$$

But

$$EU_i = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad U_{ci}L^\sim = [I \quad 0].$$

Defining

$$R := ET_1L^\sim, \tag{24}$$

we conclude that

$$\alpha = \text{dist}\left(R, \begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{H}_\infty [I \quad 0]\right). \tag{25}$$

An interesting special case occurs when T_2 has full row rank and T_3 has full column rank over the field of rational functions. Then U_i and U_{ci} are both square, so that

$$U_i^\sim = U_i^{-1}, \qquad U_{ci}^\sim = U_{ci}^{-1},$$

and R has the form

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $R_1 := U_i^{-1} T_1 U_{ci}^{-1}$. Then (25) simplifies to the expression

$$\alpha = \text{dist}\,(R_1, \mathbf{H}_\infty).$$

In this case α equals the distance in $\mathbf{L}_\infty$ from R_1 to the nearest $\mathbf{H}_\infty$-matrix.

Let us concentrate on this simpler problem of finding the distance from an $\mathbf{L}_\infty$-matrix R to the nearest $\mathbf{H}_\infty$-matrix. (The subscript on R_1 has been temporarily dropped.) Let $\mathbf{L}_2$ denote the Hilbert space of vector-valued square-integrable functions on the imaginary axis. The Hardy space $\mathbf{H}_2$ is a closed subspace of $\mathbf{L}_2$; let $\mathbf{H}_2^\perp$ denote its orthogonal complement. The $\mathbf{L}_\infty$-norm of R equals the norm of the corresponding Laurent operator on $\mathbf{L}_2$ (cf. (1)); denote this operator by M_R:

$$M_R f = Rf, \qquad f \in \mathbf{L}_2.$$

For an $\mathbf{H}_\infty$-matrix X the Laurent operator M_X leaves the subspace $\mathbf{H}_2$ of $\mathbf{L}_2$ invariant, i.e.,

$$\text{if } f \in \mathbf{H}_2 \quad \text{then } M_X f = Xf \in \mathbf{H}_2. \tag{26}$$

If $\Pi: \mathbf{L}_2 \to \mathbf{H}_2^\perp$ denotes the *orthogonal projection*, then (26) is equivalent to the condition that

$$\Pi M_X \,|\, \mathbf{H}_2 = 0.$$

Thus we have that

$$\begin{aligned} \operatorname{dist}(R, \mathbf{H}_\infty) &= \min\{\|R - X\|_\infty : X \in \mathbf{H}_\infty\} \\ &= \min\{\|M_R - M_X\| : X \in \mathbf{H}_\infty\} \\ &\geqq \min\{\|\Pi(M_R - M_X)\,|\,\mathbf{H}_2\| : X \in \mathbf{H}_\infty\} \\ &= \|\Pi M_R \,|\, \mathbf{H}_2\|. \end{aligned} \tag{27}$$

In fact, equality holds in (27).

THEOREM 5.2. *The distance from a matrix R in $\mathbf{L}_\infty$ to the nearest matrix in $\mathbf{H}_\infty$ equals the norm of the operator $\Pi M_R \,|\, \mathbf{H}_2$ from $\mathbf{H}_2$ to $\mathbf{H}_2^\perp$.*

This result is generally known as Nehari's theorem [63] and the operator $\Pi M_R \,|\, \mathbf{H}_2$ is called the *Hankel operator with symbol R* [72]. As a concrete example, consider the scalar-valued function $R(s) = (s-1)^{-1}$. For g in $\mathbf{H}_2$ we have

$$R(s)g(s) = (s-1)^{-1}g(1) + (s-1)^{-1}[g(s) - g(1)].$$

The first function on the right-hand side belongs to $\mathbf{H}_2^\perp$ and the second to $\mathbf{H}_2$. Hence the Hankel operator maps $g(s)$ in $\mathbf{H}_2$ into $(s-1)^{-1}g(1)$ in $\mathbf{H}_2^\perp$.

The Hankel operator has a time-domain version (e.g. [43]). Suppose $R(s)$ is analytic in a strip containing the imaginary axis; such would be the case if $R(s)$ were rational, for example. Taking the region of convergence to be this strip, let $r(t)$ denote the inverse bilateral Laplace transform of $R(s)$. The linear system with impulse response $r(t)$ is therefore $\mathbf{L}_2(-\infty, \infty)$-stable, but noncausal in general. The Hankel operator in the time-domain maps a function u in $\mathbf{L}_2[0, \infty)$ into a function y in $\mathbf{L}_2(-\infty, 0]$ according to the convolution equation

$$y(t) = \int_0^\infty r(t-\tau)u(\tau)\, d\tau, \qquad t \leqq 0.$$

Since the bilateral Laplace transformation is an isomorphism from $\mathbf{L}_2[0, \infty)$ onto $\mathbf{H}_2$ and from $\mathbf{L}_2(-\infty, 0]$ onto $\mathbf{H}_2^\perp$, the Hankel operators in the two domains have equal norms. A causal system leaves $\mathbf{L}_2[0, \infty)$ invariant, so its Hankel operator equals zero. Interpreted in the time-domain, Theorem 5.2 states that the distance from the noncausal system with impulse response $r(t)$ to the nearest causal system equals the norm of the Hankel operator; in other words, the Hankel operator's norm is a measure of noncausality. Here the distance is the norm of the error system considered as a mapping on $\mathbf{L}_2(-\infty, \infty)$.

It is a useful fact that the norm of a Hankel operator with a rational symbol can be computed by state-space methods. Let R be a matrix in $\mathbf{RL}_\infty$ and let $C(s-A)^{-1}B$ be a minimal realization of its antistable part, i.e.,

$$R(s) = C(s-A)^{-1}B + (\text{a matrix in } \mathbf{RH}_\infty), \tag{28}$$

and the eigenvalues of A lie in Re $s>0$. Introduce the controllability and observability gramians

$$L_c := \int_{-\infty}^{0} e^{At}BB'\, e^{A't}\, dt, \tag{29}$$

$$L_0 := \int_{-\infty}^{0} e^{A't}C'C\, e^{At}\, dt. \tag{30}$$

Thus L_c and L_o are the unique solutions of the Lyapunov equations

$$AL_c + L_cA' = BB', \tag{31}$$

$$A'L_o + L_oA = C'C. \tag{32}$$

It can be proved [81] that L_cL_o has only real, nonnegative eigenvalues.

LEMMA 5.2 (e.g. [43]). *The Hankel operator* $\Gamma := \Pi M_R | \mathbf{H}_2$ *with rational symbol* R *has finite rank. The operator* $\Gamma^*\Gamma$ *and the matrix* L_cL_o *share the same nonzero eigenvalues. In particular, the norm of* Γ *equals the square-root of the largest eigenvalue of* L_cL_o.

Now let us return to the general case. From (25) α equals the distance in $\mathbf{L}_\infty$ from R to the subspace

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{H}_\infty [I \quad 0].$$

A matrix in this subspace has the form

$$\begin{bmatrix} I \\ 0 \end{bmatrix} X [I \quad 0] = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} =: \underline{X} \tag{33}$$

for some X in $\mathbf{H}_\infty$, and the corresponding Laurent operator is $M_{\underline{X}}$. This operator acts on $\mathbf{L}_2$-vectors partitioned conformably with the partitioning in (33):

$$M_{\underline{X}} \begin{bmatrix} f \\ g \end{bmatrix} = \underline{X} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} Xf \\ 0 \end{bmatrix}.$$

Hence the domain of this operator is the product space (external direct sum) $\mathbf{L}_2 \times \mathbf{L}_2$. Moreover, since $X \in \mathbf{H}_\infty$, the subspace $\mathbf{H}_2 \times \mathbf{L}_2$ is invariant under this operator, or equivalently

$$\underline{\Pi} M_{\underline{X}} | (\mathbf{H}_2 \times \mathbf{L}_2) = 0,$$

where $\underline{\Pi}$ denotes the orthogonal projection from $\mathbf{L}_2 \times \mathbf{L}_2$ onto $\mathbf{H}_2^\perp \times \mathbf{L}_2$. As in (27) we conclude that

$$\operatorname{dist}\left(R, \begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{H}_\infty [I \quad 0] \right) \geqq \| \underline{\Pi} M_R | (\mathbf{H}_2 \times \mathbf{L}_2) \|.$$

Again, equality holds.

To recap, let R be defined as in (24), let M_R denote the Laurent operator on $\mathbf{L}_2 \times \mathbf{L}_2$ of multiplication by R, and let $\underline{\Pi}$ denote the orthogonal projection from $\mathbf{L}_2 \times \mathbf{L}_2$ onto $\mathbf{H}_2^\perp \times \mathbf{L}_2$.

THEOREM 5.3 [29]. *The minimal model-matching error* α *equals the norm of the operator*

$$\underline{\Pi} M_R | (\mathbf{H}_2 \times \mathbf{L}_2) \tag{34}$$

from $\mathbf{H}_2 \times \mathbf{L}_2$ *to* $\mathbf{H}_2^\perp \times \mathbf{L}_2$.

Operator (34) is not a Hankel operator (by definition) and at present there is unfortunately no direct procedure for computing its norm. It is easy to get crude bounds for its norm: an upper bound is $\|R\|_\infty$ and a lower bound is the maximum of

$$\|[0 \quad I]R\|_\infty, \qquad \left\| R \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_\infty.$$

5.3. Nearly optimal solutions. The value of α cannot be computed directly, as we just noted, but it is possible to compute, by iteration, an upper bound which is as close to α as desired. To do this, let α_1 and α_2 be, respectively, any lower and upper bounds for α, for example, those given at the end of the previous subsection. A number γ satisfies the inequality $\alpha < \gamma$ if and only if (see (21))

$$\text{dist}\,(T_1, T_2\mathbf{H}_\infty T_3) < \gamma. \tag{35}$$

Thus testing if (35) holds for several values of γ in the interval $[\alpha_1, \alpha_2]$ serves to locate α for any desired accuracy. A bisection search could be used.

This subsection treats the problem of checking if (35) is true for a prespecified γ and, when it is true, of finding all Q's in $\mathbf{RH}_\infty$ which achieve the inequality

$$\|T_1 - T_2QT_3\|_\infty < \gamma;$$

when γ is a bit larger than α, such Q's are nearly optimal. It will be shown that (35) is equivalent to the following three conditions:

$$\|Y\|_\infty < \gamma, \qquad \|Z\|_\infty < 1, \qquad \text{dist}\,(R, \mathbf{H}_\infty) < 1.$$

Here[1] R, Y and Z are $\mathbf{RL}_\infty$-matrices which are computed from T_i $(i = 1-3)$ and γ (Theorem 5.4 below). Notice that the distance from R to $\mathbf{H}_\infty$ can be readily computed (Theorem 5.2, Lemma 5.2).

We require two definitions. Let F be an $\mathbf{RL}_\infty$-matrix and let η be a positive number. The inequality

$$\eta > \|F\|_\infty \tag{36}$$

is equivalent to the condition

$$\eta^2 I - F(j\omega)^* F(j\omega) > 0 \quad \text{for all } 0 \leqq \omega \leqq \infty.$$

This latter condition implies that the matrix $\eta^2 I - F^\sim F$ has a spectral factorization:

$$\eta^2 I - F^\sim F = F_o^\sim F_o, \qquad F_o, F_o^{-1} \in \mathbf{RH}_\infty.$$

Such a matrix F_o will be called a *spectral factor of* $\eta^2 I - F^\sim F$. (It is outer, hence the subscript o.) The same inequality, (36), implies that

$$\eta^2 I - FF^\sim = F_{co} F_{co}^\sim$$

for some $F_{co}, F_{co}^{-1} \in \mathbf{RH}_\infty$; F_{co} will be called a *co-spectral factor of* $\eta^2 I - FF^\sim$.

Let us consider, first, condition (35) under the simplifying assumption that $T_3 = I$. Write $T_2 = U_i U_o$ where U_i is inner and U_o is outer, and let $Q \in \mathbf{RH}_\infty$. Then, as in the previous subsection, the matrices $T_1 - T_2 Q$ and

$$\begin{bmatrix} U_i^\sim \\ I - U_i U_i^\sim \end{bmatrix} (T_1 - T_2 Q) = \begin{bmatrix} U_i^\sim T_1 - U_o Q \\ (I - U_i U_i^\sim) T_1 \end{bmatrix}$$

[1] The matrix R in this subsection is defined somewhat differently from that in the previous one.

have equal norms. Defining

$$Y := (I - U_i U_i^{\sim}) T_1,$$

we obtain that

$$\|T_1 - T_2 Q\|_\infty < \gamma$$

if and only if

$$\left\| \begin{bmatrix} U_i^{\sim} T_1 - U_o Q \\ Y \end{bmatrix} \right\|_\infty < \gamma. \tag{37}$$

It can be shown that (37) holds if and only if $\|Y\|_\infty < \gamma$ and

$$\|(U_i^{\sim} T_1 - U_o Q) Y_o^{-1}\|_\infty < 1,$$

where Y_o is a spectral factor of $\gamma^2 I - Y^{\sim} Y$. We conclude that

$$\operatorname{dist}(T_1, T_2 \mathbf{H}_\infty) < \gamma$$

if and only if $\|Y\|_\infty < \gamma$ and

$$\operatorname{dist}(U_i^{\sim} T_1 Y_o^{-1}, \mathbf{H}_\infty) < 1.$$

The general result is as follows.

THEOREM 5.4 [20], [21]. *Let $Q \in \mathbf{RH}_\infty$ and $\gamma > 0$. Then*

$$\|T_1 - T_2 Q T_3\|_\infty < \gamma$$

if and only if

$$\|Y\|_\infty < \gamma, \qquad \|Z\|_\infty < 1, \qquad \|R - X\|_\infty < 1, \tag{38}$$

where R, Y, Z are $\mathbf{RL}_\infty$-matrices and X is an $\mathbf{RH}_\infty$-matrix defined as follows:

$$T_2 = U_i U_o, \quad U_i \textit{ inner}, \quad U_o \textit{ outer}, \tag{39}$$

$$Y := (I - U_i U_i^{\sim}) T_1, \tag{40}$$

$$Y_o = \textit{spectral factor of } \gamma^2 I - Y^{\sim} Y,$$

$$T_3 Y_o^{-1} = V_{co} V_{ci}, \quad V_{co} \textit{ co-outer}, \quad V_{ci} \textit{ co-inner}, \tag{41}$$

$$Z := U_i^{\sim} T_1 Y_o^{-1} (I - V_{ci}^{\sim} V_{ci}), \tag{42}$$

$$Z_{co} = \textit{co-spectral factor of } I - Z Z^{\sim}, \tag{43}$$

$$R := Z_{co}^{-1} U_i^{\sim} T_1 Y_o^{-1} V_{ci}^{\sim}, \tag{44}$$

$$X := Z_{co}^{-1} U_0 Q V_{co}. \tag{45}$$

Observe that $Z_{co}^{-1} U_o$ is right-invertible over $\mathbf{RH}_\infty$ and V_{co} is left-invertible over $\mathbf{RH}_\infty$. Thus (45) provides a linear relation between Q's satisfying (35) and X's satisfying (38).

Let us recap. Suppose the objective is to compute an upper bound γ for α such that $\gamma - \alpha$ is less than a prespecified number, and then to determine a Q in $\mathbf{RH}_\infty$ satisfying

$$\|T_1 - T_2 Q T_3\|_\infty < \gamma.$$

To accomplish this, first determine lower and upper bounds α_1 and α_2 for α. Then, select a trial value for γ in the interval $[\alpha_1, \alpha_2]$. Next, test to see if the following conditions hold:

$$\|Y\|_\infty < \gamma, \quad \|Z\|_\infty < 1, \quad \operatorname{dist}(R, \mathbf{H}_\infty) < 1.$$

If so, reduce the value of γ; if not, increase it. When a sufficiently accurate upper bound is obtained, find an X in $\mathbf{RH}_\infty$ such that $\|R-X\|_\infty<1$. Finally, solve (45) for a Q in $\mathbf{RH}_\infty$.

The part of the procedure which remains to be described is how to find such an X. This is the next topic.

5.4. Best approximation. Let R be an $\mathbf{RL}_\infty$-matrix. The problem of best approximation is that of finding one, some, or all matrices X in $\mathbf{RH}_\infty$ such that

$$\|R-X\|_\infty=\operatorname{dist}(R,\mathbf{RH}_\infty).$$

Such X's will be termed *optimal*. This problem has an extensive theory, involving several different approaches. In this paper there is space to describe only two of them.

The first approach, due to Adamjan, Arov and Krein [1], is applicable only to the special case where R and X are scalar-valued; then the optimal X is unique and $R-X$ is a scalar times an inner function (cf. Prop. 5.1). As in Lemma 5.2 let Γ denote the Hankel operator with symbol R, and consider the self-adjoint operator

$$\Gamma\Gamma^*:\mathbf{H}_2^\perp\to\mathbf{H}_2^\perp.$$

Let λ^2 denote the maximum eigenvalue of $\Gamma\Gamma^*$, let f be a corresponding eigenvector, and define $g:=\lambda^{-1}\Gamma^*f$. Observe that f and g satisfy the equations

$$\Gamma g=\lambda f,\qquad \Gamma^*f=\lambda g;$$

such vectors form what is called a Schmidt pair for Γ.

THEOREM 5.5 [1]. *The optimal X equals $R-\lambda f/g$.*

Silverman and Bettayeb [81] employed this formula together with state-space realizations to get a simple way to compute the optimal X. As in (28)-(30) let $C(s-A)^{-1}B$ be a minimal realization of the antistable part of $R(s)$ and let L_c and L_o denote the controllability and observability gramians. By Lemma 5.2 λ^2 (defined above) equals the maximum eigenvalue of L_cL_o; let w be a corresponding eigenvector and define $v:=\lambda^{-1}L_ow$.

COROLLARY 5.1 [81]. *The optimal X is given*

$$X(s)=R(s)-\lambda[A,w,C,0]/[-A',v,B',0].$$

The second approach, due to Ball and Helton [4], applies to the general matrix-valued problem. The theory of suboptimal X's is simpler than the theory of optimal ones, so a characterization will be presented of all X's in $\mathbf{RH}_\infty$ such that

$$\|R-X\|_\infty\leqq\beta,$$

where β can be any positive number greater than $\operatorname{dist}(R,\mathbf{RH}_\infty)$. To simplify notation slightly, scale R and X by the factor β^{-1}; now $\operatorname{dist}(R,\mathbf{RH}_\infty)<1$ and the problem is to find all X's in $\mathbf{RH}_\infty$ such that $\|R-X\|_\infty\leqq1$. Or, in terms of $S:=R-X$, the problem is to find all S's in $\mathbf{RL}_\infty$ such that $R-S\in\mathbf{RH}_\infty$ and $\|S\|_\infty\leqq1$.

An outline of the Ball-Helton theory takes three steps. First, instead of looking at $\mathbf{RL}_\infty$-matrices, we look at graphs of operators. Consider the restriction to $\mathbf{H}_2$ of the Laurent operator induced by R:

$$M_R\,|\,\mathbf{H}_2:\mathbf{H}_2\to\mathbf{L}_2,\qquad f\to Rf.$$

The graph of this operator, which we shall call the *graph* of R, is the set of ordered pairs (Rf, f) in $\mathbf{L}_2 \times \mathbf{H}_2$. Let us write these ordered pairs as 2-vectors $\binom{Rf}{f}$, so that the graph has the representation

$$\mathbf{G}_R := \begin{bmatrix} R \\ I \end{bmatrix} \mathbf{H}_2.$$

If $R - S \in \mathbf{RH}_\infty$ and $\|S\|_\infty \leqq 1$, what properties does the graph of S have? First, the condition $R - S \in \mathbf{RH}_\infty$ restricts the graph to being contained in a certain subspace of $\mathbf{L}_2 \times \mathbf{H}_2$, namely

$$\mathbf{W} := \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} (\mathbf{H}_2 \times \mathbf{H}_2).$$

This is easily seen as follows:

$$\begin{aligned} \mathbf{G}_S &= \begin{bmatrix} S \\ I \end{bmatrix} \mathbf{H}_2 \\ &= \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} \begin{bmatrix} S - R \\ I \end{bmatrix} \mathbf{H}_2 \\ &\subset \mathbf{W}. \end{aligned}$$

Second, the graph of S is *steep*, meaning that

$$\text{if } \begin{pmatrix} f \\ g \end{pmatrix} \in \mathbf{G}_S, \quad \text{then } \|f\|_2 \leqq \|g\|_2. \tag{46}$$

(The slope of the line in the plane through the origin and the point $(\|f\|_2, \|g\|_2)$ is at least 45 degrees.) Fact (46) follows immediately from the condition $\|S\|_\infty \leqq 1$. These two properties characterize the graph.

LEMMA 5.3 [4]. *Let S be an $\mathbf{RL}_\infty$-matrix (of the same dimensions as R). Then $\|S\|_\infty \leqq 1$ and $R - S \in \mathbf{RH}_\infty$ if and only if the graph of S is steep and is contained in* $\mathbf{W}$.

The second step in the theory is to represent $\mathbf{W}$ in a way which is useful for characterizing steep graphs. For this representation introduce the matrix

$$J := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

(The dimension of the upper left unit matrix equals the number of rows of R; the dimension of the lower right, the number of columns.) A square matrix L in $\mathbf{RL}_\infty$ (of appropriate dimensions) is *J-unitary* if $L^\sim J L = J$.

LEMMA 5.4 [4]. *There exists a J-unitary matrix L such that* $L(\mathbf{H}_2 \times \mathbf{H}_2) = \mathbf{W}$.

Lemma 5.4 is a generalization of Beurling's theorem [46]. The useful fact about L is that, because it is J-unitary, it maps a steep subspace of $\mathbf{H}_2 \times \mathbf{H}_2$ into a steep subspace of $\mathbf{W}$ (in fact, it is a one-to-one correspondence between such subspaces). Thus, if $Y \in \mathbf{RH}_\infty$ and $\|Y\|_\infty \leqq 1$, then $L\mathbf{G}_Y$ is a steep subspace of $\mathbf{W}$.

The third step in the theory is to combine Lemmas 5.3 and 5.4 to obtain the main result, which is stated in terms of X.

THEOREM 5.6 [4]. *The set of all X's in $\mathbf{RH}_\infty$ such that $\|R - X\|_\infty \leqq 1$ is parametrized by the formula*

$$X = R - X_1 X_2^{-1}, \qquad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = L \begin{bmatrix} Y \\ I \end{bmatrix},$$

$$Y \in \mathbf{RH}_\infty, \qquad \|Y\|_\infty \leqq 1.$$

Ball and Ran [5], [6] showed how to obtain a state-space realization of L. A summary of their procedure is as follows. As above, let $[A, B, C, 0]$ be a minimal realization of the antistable part of $R(s)$ and let L_c and L_o denote the controllability and observability gramians. (Recall that R has been scaled so that its distance to $\mathbf{RH}_\infty$ is less than 1.) Then a realization of L is

$$L(s) = [\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, I],$$

where

$$\underset{\sim}{A} := \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix},$$

$$\underset{\sim}{B} := \begin{bmatrix} -L_c & I \\ I & -L_o \end{bmatrix} \begin{bmatrix} I - L_o L_c & 0 \\ 0 & I - L_c L_o \end{bmatrix}^{-1} \begin{bmatrix} C' & 0 \\ 0 & B \end{bmatrix},$$

$$\underset{\sim}{C} := \begin{bmatrix} C & 0 \\ 0 & -B' \end{bmatrix}.$$

Other approaches to the best approximation problem are those of Glover [43] (based on [2]) and Chang and Pearson [8] (based on [16]).

6. A numerical example. The purpose of this section is to elucidate the theory of §§ 3–5 by carrying out a numerical example of the tracking problem of § 2. With reference to Fig. 5 we take the unstable nonminimum phase plant

$$P(s) = \frac{s-1}{s(s-2)}.$$

The weighting factor ρ in (6) and the weighting filter W in Fig. 5 are as follows:

$$\rho = 1, \qquad W(s) = \frac{s+1}{10s+1}.$$

The Bode magnitude plot of W is nearly 0 db up to the frequency .1, so this choice of W reflects a family of reference signals having their energy concentrated in the frequency band $[0, .1]$. With this choice of P, W and ρ, the transfer matrix G in Fig. 2 is determined from (7) to be

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \dfrac{s+1}{10s+1} & -\dfrac{s-1}{s(s-2)} \\ 0 & 1 \\ \hline \dfrac{s+1}{10s+1} & 0 \\ 0 & \dfrac{s-1}{s(s-2)} \end{array}\right].$$

The first step in computing a controller is to obtain the matrices T_i $(i = 1-3)$ in the equivalent model-matching problem. We shall use the state-space method of § 4 (although this is not the easiest way for this simple example). We begin with a minimal realization of G:

$$G(s)=[A,\quad B,\quad C,\quad D],$$

$$A=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},\qquad B=[B_1\quad B_2]=\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{array}\right],$$

$$C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\left[\begin{array}{ccc} .09 & -.5 & -.5 \\ 0 & 0 & 0 \\ \hline .09 & 0 & 0 \\ 0 & .5 & .5 \end{array}\right],\qquad D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}=\left[\begin{array}{c|c} .1 & 0 \\ 0 & 1 \\ \hline .1 & 0 \\ 0 & 0 \end{array}\right].$$

Now F and H are chosen so that the matrices

$$A_F := A+B_2F,\qquad A_H := A+HC_2$$

are stable. The exact locations of the eigenvalues are not important for the purpose at hand; the choice

$$F=[0\quad .5\quad -4.5],\qquad H=\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -9 \end{bmatrix}$$

yields

$$A_F=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & .5 & -4.5 \\ 0 & .5 & -2.5 \end{bmatrix},\qquad A_H=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & .5 & .5 \\ 0 & -4.5 & -2.5 \end{bmatrix},$$

which both have spectrum $\{-.1, -1, -1\}$. Then (19) produces

$$T_1(s)=T_3(s)=\begin{bmatrix} \dfrac{s+1}{10s+1} \\ 0 \end{bmatrix},\qquad T_2(s)=\begin{bmatrix} -\dfrac{s-1}{(s+1)^2} \\ \dfrac{s(s-2)}{(s+1)^2} \end{bmatrix}.$$

Theorem 5.1 can now be used to show that there does indeed exist an optimal proper controller: $T_2(j\omega)$ and $T_3(j\omega)$ both have rank 1 for all $0\leqq\omega\leqq\infty$.

The second step is to compute an upper bound γ for the infimal model-matching error α defined in (20). Recall from Theorem 5.4 that $\gamma>\alpha$ if and only if the following three conditions hold:

$$\|Y\|_\infty<\gamma,\qquad \|Z\|_\infty<1,\qquad \operatorname{dist}(R, H_\infty)<1. \tag{47}$$

The matrices Y, Z and R are computed as in Theorem 5.4. A simple way to do the inner-outer factorization (39) of T_2 is first to get

$$T_2^\sim(s)T_2(s)=\frac{s^4-5s^2+1}{(-s+1)^2(s+1)^2}.$$

Solving $T_2^\sim T_2=U_o^\sim U_o$ for an outer function U_o gives

$$U_o(s)=\frac{s^2+\sqrt{7}s+1}{(s+1)^2}.$$

Then the inner factor U_i is

$$U_i = T_2 U_o^{-1}, \qquad U_i(s) = \frac{1}{s^2+\sqrt{7}s+1}\begin{bmatrix} -s+1 \\ s(s-2) \end{bmatrix}.$$

The matrix Y defined in (40) is determined next:

$$Y(s) = \frac{s+1}{(10s+1)(s^4-5s^2+1)}\begin{bmatrix} s^2(s^2-4) \\ -s(s+1)(s-2) \end{bmatrix}. \tag{48}$$

From (47) the value of γ must be at least $\|Y\|_\infty$. From (48) we get

$$Y^\sim(s)Y(s) = \frac{s^2(s^2-4)(s^2-1)}{(100s^2-1)(s^4-5s^2+1)}.$$

The spectral factor of $Y^\sim Y$ is

$$\frac{s(s+2)(s+1)}{(10s+1)(s^2+\sqrt{7}s+1)}. \tag{49}$$

Thus $\|Y\|_\infty$ equals the $\mathbf{H}_\infty$-norm of the function (49), which can be read off its Bode magnitude plot. This yields $\|Y\|_\infty = .1683$. Thus γ must be at least .1683. It turns out that for $\gamma = .2$ the distance from R to $\mathbf{H}_\infty$ is greater than 1, violating (47). We shall show that (47) holds for $\gamma = .3$.

The spectral factor of $\gamma^2 - Y^\sim Y$ is

$$Y_o(s) = \frac{2.828s^3+7.615s^2+3.165s+.3}{(10s+1)(s^2+\sqrt{7}s+1)}.$$

Since $T_3 Y_o^{-1}$ is already co-outer, in (41) we can take $V_{co} = T_3 Y_o^{-1}$ and $V_{ci} = 1$. Then from (42) $Z = 0$, and from (43) $Z_{co} = 1$. Finally, from (44) we get

$$R(s) = \frac{(s+1)^2(s^2+\sqrt{7}s+1)}{(s^2-\sqrt{7}s+1)(2.828s^3+7.615s^2+3.165s+.3)}.$$

We shall compute the distance from R to $\mathbf{H}_\infty$ using Theorem 5.2 and Lemma 5.2. The antistable part of R is

$$\frac{.9267}{s-2.189} - \frac{.8217}{s-.4569},$$

which has the realization

$$A = \begin{bmatrix} 2.189 & 0 \\ 0 & .4569 \end{bmatrix}, \qquad B = \begin{bmatrix} .9267 \\ -.8217 \end{bmatrix}, \qquad C = [1 \quad 1].$$

The Lyapunov equations (31) and (32) are readily solved, giving

$$L_c = \begin{bmatrix} .1962 & -.2878 \\ -.2878 & .7389 \end{bmatrix}, \qquad L_o = \begin{bmatrix} .2284 & .3780 \\ .3780 & 1.094 \end{bmatrix}.$$

The distance equals the square root of the largest eigenvalue of $L_c L_o$:

$$\operatorname{dist}(R, \mathbf{H}_\infty) = .7907.$$

Thus (47) is satisfied for $\gamma = .3$ and we conclude that $.2 < \alpha < .3$. We could at this point find a better estimate for α by reducing γ and checking (47) again, but we shall instead complete the computation with $\gamma = .3$.

The third step is to find the closest $\mathbf{H}_\infty$-function X to R. Since these functions are scalar-valued, we can use Proposition 5.2. We already have that λ^2 $(=.7907^2)$ equals the maximum eigenvalue of L_cL_o; a corresponding eigenvector is

$$w=[1 \quad -2.862]'.$$

Then $v:=\lambda^{-1}L_o w$ equals

$$[-1.079 \quad -3.483]'.$$

The formula in Proposition 5.2 yields

$$X(s)=\frac{(s^2+\sqrt{7}s+1)(.7170s^2+1.912s+.7628)}{(.3206s+1)(2.828s^3+7.615s^2+3.165s+.3)}.$$

The fourth step is to solve (45) for a Q in $\mathbf{RH}_\infty$. The matrix Q has dimensions 1×2:

$$Q=[Q_1 \quad Q_2].$$

Equation (45) determines Q_1 uniquely:

$$Q_1(s)=\frac{(s+1)(.7170s^2+1.912s+.7628)}{(.3206s+1)(s^2+\sqrt{7}s+1)},$$

whereas Q_2 is unconstrained, and hence may be taken to be zero.

Finally, the controller K is computed using Theorem 3.1. We can determine M_2, N_2, X_2 and Y_2 from (14), (15), (17), (18) and then get K from (10):

$$K=[C_1 \quad C_2],$$

$$C_1(s)=-\frac{(s+1)^3(.7170s^2+1.912s+.7628)}{(.3206s+1)(s^2+\sqrt{7}s+1)(s^2+6s-23)},$$

$$C_2(s)=-\frac{41s-1}{s^2+6s-23}.$$

(The reader will have noticed that C_1 is unstable, so the controller cannot be implemented as shown in Fig. 5. However, the theory guarantees that C_2 contains the unstable factor of C_1. This common unstable factor would be moved past the summing junction into the loop.)

The properties of this design are illustrated in the Bode magnitude plots of Fig. 7. The transfer function, say H_1, from reference r to tracking error $r-v$ has magnitude less than -10 db over the frequency band $[0, .1]$ of r (smaller tracking error could be obtained by reducing the weighting ρ on control energy), it peaks to about 4 db outside the operating band, and it rolls off to 0 db at high frequency, as it must for a proper controller. This sort of shape is characteristic of $\mathbf{H}_\infty$ designs. The transfer function, say H_2, from r to u has a zero at $s=0$ because P has a pole there. The actual quantity being minimized in this example is the $\mathbf{H}_\infty$-norm of the transfer matrix

$$\begin{bmatrix} H_1 W \\ H_2 W \end{bmatrix}$$

from w to $(r-v, u)'$. This norm equals the supremum of

$$(|H_1(j\omega)|^2+|H_2(j\omega)|^2)^{1/2}|W(j\omega)| \tag{50}$$

over all ω. For our design the function (50) is very nearly flat at -12 db (the supremum of (50) must be greater than -14 db corresponding to the bound $\alpha>.2$).

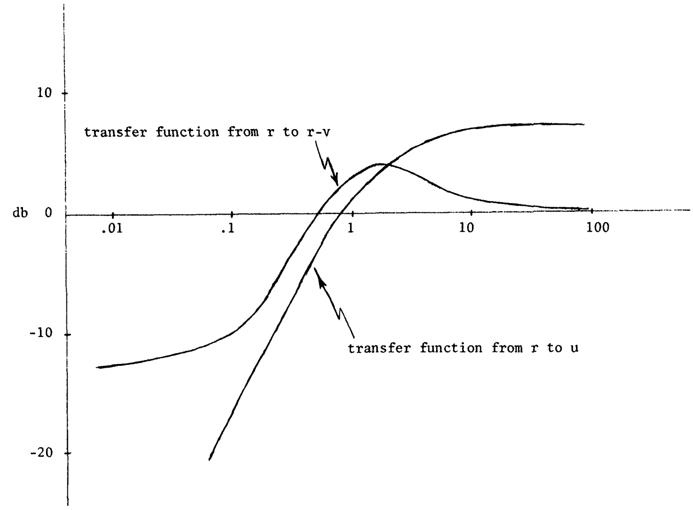

FIG. 7. *Bode magnitude plots.*

7. Achievable performance. For some simple examples of the standard problem it is possible to obtain useful bounds on achievable performance, sometimes even to characterize achievable performance exactly. Such results can shed light on what properties of a system affect its performance. This section presents three illustrative examples.

Figure 8 shows a feedback system with a disturbance signal w referred to the output of the plant P. As usual, P is strictly proper and K is proper. The transfer matrix from w to y is the *sensitivity matrix* $S := (I - PK)^{-1}$.

Suppose first that the spectrum of w is confined to a prespecified interval of frequencies $[-\omega_1, \omega_1]$, $\omega_1 > 0$. Then the problem of attenuating the effect of w on the output y of the plant is equivalent to the problem of making $\sigma_{\max}[S(j\omega)]$ uniformly small on the interval $[-\omega_1, \omega_1]$. Let χ denote the characteristic function of this interval, i.e.,

$$\chi(j\omega) = 1, \quad |\omega| \leqq \omega_1, \qquad \chi(j\omega) = 0, \quad |\omega| > \omega_1.$$

Then the maximum value of $\sigma_{\max}[S(j\omega)]$ over the interval $[-\omega_1, \omega_1]$ equals the $\mathbf{L}_\infty$-norm $\|\chi S\|_\infty$. It may happen that as we try to make $\|\chi S\|_\infty$ smaller and smaller, the global bound $\|S\|_\infty$ becomes larger and larger. This is unpleasant because a large value of $\|S\|_\infty$ means the system has poor stability margin. (Think of the scalar-valued case: if $\|S\|_\infty$ is large, then the Nyquist plot of PK passes near the critical point.)

The first result says that if P is minimum phase, then $\|\chi S\|_\infty$ can be made as small as desired while $\|S\|_\infty$ is simultaneously maintained less than any bound δ. Of course δ must be greater than unity since $\|S\|_\infty \geqq 1$ for every stabilizing K.

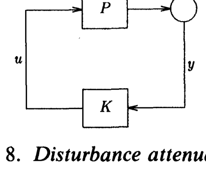

FIG. 8. *Disturbance attenuation.*

THEOREM 7.1 [67], [99]. *If P has a right-inverse which is analytic in* $\operatorname{Re} s \geqq 0$, *then for every $\varepsilon > 0$ and $\delta > 1$ there exists a stabilizing K such that*

$$\|\chi S\|_\infty < \varepsilon, \qquad \|S\|_\infty < \delta.$$

On the other hand, if P has a zero in the right half-plane, then $\|S\|_\infty$ must necessarily increase without limit if $\|\chi S\|_\infty$ tends to zero. This might be described as the "waterbed effect."

THEOREM 7.2 [33], [35], [39]. *If at some point in* $\operatorname{Re} s > 0$ *the rank of P is less than the number of its rows, then there exists a positive real number a such that for every stabilizing K*

$$\|\chi S\|_\infty \|S\|_\infty^a > 1.$$

For the third result consider, with regard to Fig. 8 again, the problem of attenuating the effect of w (no longer restricted to be bandlimited) on the control signal u; that is, the problem is to achieve feedback stability by a controller which limits as much as possible the control effort. The transfer matrix from w to u equals KS, so the objective is to minimize $\|KS\|_\infty$. The case where P is stable is trivial: an optimal K is $K = 0$. So we suppose P is not stable. For technical reasons it is assumed that P has no poles on the imaginary axis; thus P belongs to $\mathbf{RL}_\infty$ but not $\mathbf{RH}_\infty$. Let Γ denote the Hankel operator with symbol P and let $\sigma_{\min}(\Gamma)$ denote the smallest (nonzero) singular value of Γ.

THEOREM 7.3 [44], [87]. *If P belongs to $\mathbf{RL}_\infty$ but not $\mathbf{RH}_\infty$, then the minimum value of $\|KS\|_\infty$ over all stabilizing K's equals the reciprocal of $\sigma_{\min}(\Gamma)$.*

8. Comparison with the Wiener–Hopf approach. In the model-matching problem posed in § 2 and solved in § 5, the criterion is to minimize the $\mathbf{H}_\infty$-norm of the error transfer matrix $T_1 - T_2QT_3$. It is evident from § 5 that, at least in the matrix case, optimal Q's are not very easy to compute. By way of contrast the Wiener–Hopf approach to the model-matching problem is to minimize the $\mathbf{H}_2$-norm of the error transfer matrix. (In Fig. 4, if w is standard white noise, then the root-mean-square value of z equals the $\mathbf{H}_2$-norm of the transfer matrix from w to z.) It is relatively easy to compute optimal Q's for the $\mathbf{H}_2$-criterion. Therefore it is perhaps legitimate to ask if the computational effort required for the $\mathbf{H}_\infty$ approach is worthwhile. How much better is the $\mathbf{H}_\infty$ solution than the $\mathbf{H}_2$ solution?

To give one possible answer to this question we consider for simplicity the case where T_i $(i = 1-3)$ are scalar-valued, in which case we may assume that $T_3 = 1$ by redefining T_2. The function T_2 is assumed not to be zero anywhere on the extended imaginary axis (so that the following two optima exist). Let Q_2 denote the optimal solution for the $\mathbf{H}_2$ criterion

$$\|T_1 - T_2Q\|_2 = \text{minimum},$$

and let Q_∞ denote the optimal solution for the $\mathbf{H}_\infty$ criterion

$$\|T_1 - T_2Q\|_\infty = \text{minimum}.$$

If we were to use the $\mathbf{H}_2$ solution, the supremal value of $\|z\|_2$ over all $\|w\|_2 \leqq 1$ would equal $\|T_1 - T_2Q_2\|_\infty$. Thus the above question can be rephrased as follows: How large can the ratio

$$\|T_1 - T_2Q_2\|_\infty / \|T_1 - T_2Q_\infty\|_\infty \tag{51}$$

be?

Let k denote the number of zeros of T_2 in the right half-plane.

PROPOSITION 8.1. *The supremum of the ratio* (51) *equals* $2k$.

Here the supremum is over all T_i's in $\mathbf{RH}_\infty$ such that T_2 has k zeros in Re $s>0$ (and no zeros on the extended imaginary axis). The idea of the proof of Proposition 8.1 is as follows. We may suppose without loss of generality that T_2 is an inner function: just absorb the outer factor into Q. Then for Q in $\mathbf{H}_2$ the $\mathbf{H}_2$-norm of $T_1 - T_2Q$ equals the $\mathbf{L}_2$-norm of $T_1T_2^{-1} - Q$. Hence Q_2 equals the projection of $T_1T_2^{-1}$ onto $\mathbf{H}_2^\perp$. Denote this projection by T. Thus the $\mathbf{H}_\infty$-norm of $T_1 - T_2Q_2$ equals the $\mathbf{L}_\infty$-norm of T. Similarly, the $\mathbf{H}_\infty$-norm of $T_1 - T_2Q_\infty$ equals the norm of the Hankel operator with symbol T. Denote this operator by Γ_T. The function T has at most k poles in Re $s>0$. Glover proved ([43, Cor. 9.3]) that

$$\|T\|_\infty / \|\Gamma_T\| \leqq 2k,$$

and Jonckheere et al. [52] showed that this bound could be approached as closely as desired by suitable choice of T.

We conclude that the improvement in the $\mathbf{H}_\infty$ solution over the $\mathbf{H}_2$ solution, for the $\mathbf{H}_\infty$ criterion, can be arbitrarily great.

9. Summary. The standard problem, which includes the robust stabilization problem, is to minimize the $\mathbf{H}_\infty$-norm of the closed-loop transfer function for a fixed known plant. Under parametrization of the controller, the standard problem reduces to one of model-matching: minimize the $\mathbf{H}_\infty$-norm of an affine function. The model-matching problem reduces in turn to a sequence of best approximation problems: approximate in $\mathbf{L}_\infty$-norm an unstable transfer function by a stable one.

10. Conclusion. The standard problem is well understood and software exists for its solution [12], [21]; the software uses standard routines (such as singular-value decompositions and solving Lyapunov equations). The $\mathbf{H}_\infty$ problem with plant uncertainty, how to achieve frequency-domain performance specifications in the face of plant uncertainty, is a current area of research.

REFERENCES

[1] V. M. ADAMJAN, D. Z. AROV AND M. G. KREIN, *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem*, Math. USSR-Sb., 15 (1971), pp. 31-73.

[2] ———, *Infinite block Hankel matrices and related extension problems*, Amer. Math. Soc. Transl., 111 (1978), pp. 133-156.

[3] A. C. ANTOULAS, *A new approach to synthesis problems in linear system theory*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 465-473.

[4] J. A. BALL AND J. W. HELTON, *A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory*, J. Operator Theory, 9 (1983), pp. 107-142.

[5] J. A. BALL AND A. C. M. RAN, *Optimal Hankel norm model reductions and Wiener-Hopf factorizations I: the canonical case*, Tech. Report, Dept. of Mathematics, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, 1985.

[6] ———, *Hankel norm approximation of a rational matrix function in terms of its realization*, Tech. Report, Dept. of Mathematics, Virginia Tech., 1985.

[7] S. BOYD AND C. A. DESOER, *Subharmonic functions and performance bounds on linear time-invariant feedback systems*, Memo. No. UCB/ERL M84/51, Electronic Research Lab., Univ. of California, Berkeley, CA, 1984.

[8] B. C. CHANG AND J. B. PEARSON, *Optimal disturbance reduction in linear multivariable systems*, IEEE Trans. Automat. Control, AC-29 (1984), pp. 880-887.

[9] B. S. CHEN, *Controller synthesis of optimal sensitivity: multivariable case*, Proc. IEEE part D, 131 (1984), pp. 47-51.

[10] M. J. CHEN AND C. A. DESOER, *Necessary and sufficient condition for robust stability of linear distributed feedback systems*, Internat. J. Control, 35 (1982), pp. 255-267.

[11] L. CHENG AND J. B. PEARSON, *Frequency domain synthesis of multivariable linear regulators*, IEEE Trans. Automat. Control, AC-23 (1978), pp. 3-15.

[12] C. C. CHU, *H_∞ optimization and robust multivariable control*, Ph.D. thesis, Dept. of Electrical Engineering, Univ. of Minnesota, Minneapolis, MN, 1985.

[13] C. C. CHU AND J. C. DOYLE, *The general distance problem in H_∞ synthesis*, Proc. Conference on Decision and Control, 1985.

[14] R. F. CURTAIN AND K. GLOVER, *Robust stabilization of infinite-dimensional systems by finite-dimensional controllers*, Systems Control Lett., to appear.

[15] ———, *Robust stabilization of infinite-dimensional systems by finite-dimensional controllers: derivations and examples*, Proc. Symposium on Math. Theory of Networks and Systems, Stockholm, 1984.

[16] P. DELSARTE, Y. GENIN AND Y. KAMP, *The Nevanlinna-Pick problem for matrix-valued functions*, SIAM J. Appl. Math., 36 (1979), pp. 47-61.

[17] C. A. DESOER AND M. VIDYASAGAR, *Feedback systems: input-output properties*, Academic Press, New York, 1975.

[18] C. A. DESOER, R. W. LIU, J. MURRAY AND R. SAEKS, *Feedback system design: the fractional representation approach*, IEEE Trans. Automat. Control, AC-25 (1980), pp. 399-412.

[19] J. C. DOYLE AND G. STEIN, *Multivariable feedback design: concepts for a classical modern synthesis*, IEEE Trans. Automat. Control, AC-26 (1981), pp. 4-16.

[20] J. C. DOYLE, *Synthesis of robust controllers and filters*, Proc. Conference on Decision and Control, 1983.

[21] ———, *Lecture Notes in Advances in Multivariable Control*, Office of Naval Research/Honeywell Workshop, Minneapolis, MN, 1984.

[22] ———, *Robust stability with structured perturbations*, Proc. Symposium on Math. Theory of Networks and Systems, Stockholm, 1985.

[23] J. C. DOYLE AND C. C. CHU, *Matrix interpolation and H_∞ performance bounds*, Proc. American Control Conf., 1985.

[24] J. C. DOYLE, *Structured uncertainty in control systems*, IFAC Workshop on Model Error Concepts and Compensation, Boston, MA, 1985.

[25] P. L. DUREN, *Theory of H_p Spaces*, Academic Press, New York, 1970.

[26] A. FEINTUCH, P. KHARGONEKAR AND A. TANNENBAUM, *On the sensitivity minimization problem for linear time-varying systems*, this Journal, to appear.

[27] A. FEINTUCH AND B. A. FRANCIS, *Uniformly optimal control of linear time-varying systems*, Systems Control Lett., 5 (1984), pp. 67-71.

[28] A. FEINTUCH AND A. TANNENBAUM, *Gain optimization for distributed systems*, Systems Control Lett., 6 (1986), pp. 295-302.

[29] A. FEINTUCH AND B. A. FRANCIS, *Uniformly optimal control of linear systems*, Automatica, 21 (1986), pp. 563-574.

[30] D. S. FLAMM AND S. K. MITTER, *Progress on H_∞ optimal sensitivity for delay systems*, Tech. Report LIDS-P-1513, Massachusetts Inst. of Technology, Cambridge, MA, 1985.

[31] C. FOIAS, A. TANNENBAUM AND G. ZAMES, *Weighted sensitivity minimization for delay systems*, Tech. Report, Dept. of Electrical Engineering, McGill Univ., Montreal, 1985.

[32] Y. K. FOO AND I. POSTLETHWAITE, *An H_∞-minimax approach to the design of robust control systems*, Systems Control Lett., 5 (1984), pp. 81-88.

[33] B. A. FRANCIS, *Notes on H_∞-optimal linear feedback systems*, 1983, lectures given at Linkoping Univ., Linkoping, Sweden.

[34] B. A. FRANCIS AND G. ZAMES, *Design of H_∞-optimal multivariable feedback systems*, Proc. Conference on Decision and Control, 1983.

[35] ———, *On H_∞-optimal sensitivity theory for siso feedback systems*, IEEE Trans. Automat. Control, AC-29 (1984), pp. 9-16.

[36] B. A. FRANCIS, J. W. HELTON AND G. ZAMES, *H_∞-optimal feedback controllers for linear multivariable systems*, IEEE Trans. Automat. Control, 29 (1984), pp. 888-900.

[37] B. A. FRANCIS, *Optimal disturbance attenuation with control weighting*, in Lecture Notes in Control and Information Sci., Vol. 66, Springer-Verlag, Berlin, New York, 1985, Proc. 1984 Twente Workshop on Systems and Optimization.

[38] J. FREUDENBERG, *Issues in frequency domain feedback control*, Ph.D. thesis, Dept. of Electrical Engineering, Univ. of Illinois, Urbana, IL, 1985.

[39] J. FREUDENBERG AND D. LOOZE, *Right half-plane poles and zeros and design trade-offs in feedback systems*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 555-565.

[40] ———, *An analysis of H_∞-optimization design methods*, IEEE Trans. Automat. Control, to appear.

[41] C. GANESH AND J. B. PEARSON, *Design of optimal control systems with stable feedback*, Tech. Report #8514, Dept. of Electrical Engineering, Rice Univ., Houston, TX, 1985.

[42] T. GEORGIOU AND P. KHARGONEKAR, *A constructive algorithm for sensitivity optimization of periodic systems*, this Journal, 25 (1987), pp. 334-340.

[43] K. GLOVER, *All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds*, Internat. J. Control, 39 (1984), pp. 1115-1193.

[44] ———, *Robust stabilization of linear multivariable systems: relations to approximation*, Internat. J. Control, 43 (1986), pp. 741-766.

[45] M. J. GRIMBLE, *Optimal H_∞ robustness and the relationship to LQG design problems*, Tech. Report ICU/55, ICU, Univ. of Strathclyde, Glasgow, 1984.

[46] P. R. HALMOS, *A Hilbert Space Problem Book*, Springer-Verlag, Berlin, New York, 1982.

[47] J. W. HELTON, *Operator theory and broadband matching*, Proc. Allerton Conf., 1976.

[48] ———, *Worst case analysis in the frequency-domain: an H_∞ approach to control*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 1154-1170.

[49] ———, *Lecture notes*, NSF-CBMS Conf. on Optimization in Operator Theory, Analytic Function Theory and Electrical Engineering, Lincoln, NB, 1985.

[50] J. W. HELTON AND D. F. SCHWARTZ, *A primer on the H_∞ disk method in frequency-domain design: control*, Tech. Report, Dept. of Mathematics, Univ. of California, San Diego, CA, 1985.

[51] E. JONCKHEERE AND M. VERMA, *A spectral characterization of H_∞-optimal feedback performance: the multivariable case*, Tech. Report, Dept. of Electrical Engineering, Univ. of Southern California, Los Angeles, CA, 1986.

[52] E. A. JONCKHEERE, M. G. SAFONOV AND L. M. SILVERMAN, *Topology induced by the Hankel norm in the space of transfer matrices*, Proc. Conference on Decision and Control, 1981.

[53] T. KAILATH, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[54] P. KHARGONEKAR AND K. POOLLA, *Robust stabilization of distributed systems*, Automatica, to appear.

[55] P. KHARGONEKAR, K. POOLLA AND A. TANNENBAUM, *Robust control of linear time-invariant plants using periodic compensation*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 1088-1098.

[56] P. KHARGONEKAR AND A. TANNENBAUM, *Noneuclidean metrics and the robust stabilization of systems with parameter uncertainty*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 1005-1013.

[57] P. KHARGONEKAR AND K. POOLLA, *Uniformly optimal control of linear time-invariant plants: nonlinear time-varying controllers*, Systems Control Lett., 6 (1986), pp. 303-308.

[58] H. KIMURA, *Robust stabilization for a class of transfer functions*, IEEE Trans. Automat. Control, AC-29 (1984), pp. 788-793.

[59] S. Y. KUNG AND D. W. LIN, *Optimal Hankel-norm model reductions: multivariable systems*, IEEE Trans. Automat. Control, AC-26 (1981), pp. 832-852.

[60] H. KWAKERNAAK, *Minimax frequency-domain optimization of multivariable linear feedback systems*, IFAC World Congress, Budapest, 1984.

[61] ———, *A polynomial approach to minimax frequency domain optimization of multivariable systems*, Tech. Rept. 529, Dept. Appl. Math., Twente Univ. Tech., Twente, 1985.

[62] ———, *Minimax frequency domain performance and robustness optimization of linear feedback systems*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 994-1004.

[63] Z. NEHARI, *On bounded bilinear forms*, Ann. of Math. (2), 65 (1957), pp. 153-162.

[64] C. N. NETT, C. A. JACOBSON AND M. J. BALAS, *A connection between state-space and doubly coprime fractional representations*, IEEE Trans. Automat. Control, AC-29 (1984), pp. 831-832.

[65] C. N. NETT, *Algebraic aspects of linear control system stability*, Proc. Conference on Decision and Control, 1985.

[66] S. O'YOUNG, *Performance trade-offs in the design of multivariable controllers*, Ph.D. thesis, Dept. of Electrical Engineering, Univ. of Waterloo, Waterloo, Ontario, Canada, 1985.

[67] S. O'YOUNG AND B. A. FRANCIS, *Sensitivity trade-offs for multivariable plants*, IEEE Trans. Automat. Control, AC-30 (1985), pp. 625-632.

[68] ———, *Optimal performance and robust stabilization*, Automatica, 1986, to appear.

[69] A. PASCOAL AND P. KHARGONEKAR, *Remarks on weighted sensitivity minimization*, Tech. Report, Dept. of Electrical Engineering, Univ. of Minnesota, Minneapolis, MN, 1986.

[70] L. PERNEBO, *An algebraic theory for the design of controllers for linear multivariable systems, parts I & II*, IEEE Trans. Automat. Control, AC-26 (1981), pp. 171-194.

[71] I. POSTLETHWAITE AND Y. K. FOO, *All solutions, all-pass form solutions, and the "best" solutions to an H_∞ optimization problem in robust control*, Symposium on Math. Theory of Networks and Systems, Stockholm, 1985.

[72] S. C. POWER, *Hankel Operators on Hilbert Space*, Pitman, London, 1982.

[73] M. SAFONOV AND M. ATHANS, *A multiloop generalization of the circle criterion for stability margin analysis*, IEEE Trans. Automat. Control, AC-26 (1981), pp. 415-422.

[74] M. G. SAFONOV AND B. S. CHEN, *Multivariable stability margin optimization with decoupling and output regulation*, Proc. IEE, Part D, 129 (1982), pp. 276-282.

[75] M. G. SAFONOV, *L_∞-optimal sensitivity versus stability margin*, Proc. Conference on Decision and Control, 1983.

[76] ———, *Optimal diagonal scaling for infinity norm optimization*, Proc. American Control Conference, Boston, MA, 1985.

[77] M. G. SAFONOV AND M. S. VERMA, *L_∞ sensitivity optimization and Hankel approximation*, IEEE Trans. Automat. Control, AC-30 (1985), 279–280.

[78] I. W. SANDBERG, *An observation concerning the application of the contraction mapping fixed-point theorem and a result concerning the norm-boundedness of solutions of nonlinear functional equations*, Bell System Tech. J, 44 (1965), pp. 1809–1812.

[79] D. SARASON, *Generalized interpolation in H_∞*, Trans. Amer. Math. Soc., 127 (1967), pp. 179–203.

[80] A. SIDERIS AND M. G. SAFONOV, *Design of linear control systems for robust stability and performance*, Proc. IFAC Workshop on Model Error Concepts and Compensation, Boston, MA, 1985.

[81] L. SILVERMAN AND M. BETTAYEB, *Optimal approximation of linear systems*, Proc. Joint Automat. Control Conf., 1980.

[82] B. SZ.-NAGY AND C. FOIAS, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, New York, 1970.

[83] A. TANNENBAUM, *On the blending problem and parameter uncertainty in control theory*, Tech. Report, Dept. of Mathematics, Weizmann Inst. of Science, Rehovot, Israel, 1977.

[84] ———, *Feedback stabilization of linear dynamical plants with uncertainty in the gain factor*, Internat. J. Control, 32 (1980), pp. 1–16.

[85] ———, *Modified Nevanlinna-Pick interpolation of linear plants with uncertainty in the gain factor*, Internat. J. Control, 36 (1982), pp. 331–336.

[86] M. VERMA AND E. JONCKHEERE, *L_∞-compensation with mixed sensitivity as a broadband matching problem*, Systems Control Lett., 4 (1984), pp. 125–130.

[87] M. S. VERMA, *Synthesis of infinity-norm optimal linear feedback systems*, Ph.D. thesis, Dept. of Electrical Engineering, Univ. of Southern California, Los Angeles, CA, 1985.

[88] M. VIDYASAGAR AND H. KIMURA, *Robust controllers for uncertain linear multivariable systems*, Automatica, to appear.

[89] M. VIDYASAGAR, *The graph metric for unstable plants and robustness estimates for feedback stability*, IEEE Trans. Automat. Control, AC-29 (1984), pp. 403–418.

[90] ———, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

[91] ———, *Filtering and robust regulation using a two-parameter controller*, Proc. Symposium on Math. Theory of Networks and Systems, Stockholm, 1985.

[92] Z. Z. WANG AND J. B. PEARSON, *Regulation and optimal error reduction in linear multivariable systems*, Proc. IFAC World Congress, Budapest, 1984.

[93] D. C. YOULA, H. A. JABR AND J. J. BONGIORNO JR., *Modern Wiener-Hopf design of optimal controllers: part II*, IEEE Trans. Automat. Control, AC-21 (1976), pp. 319–338.

[94] G. ZAMES, *Nonlinear operators for system analysis*, Tech. Report 370, Research Lab. of Electronics, Massachusetts Inst. of Technology, Cambridge, MA, 1960.

[95] ———, *On the input-output stability of nonlinear time-varying feedback systems, parts* I & II, IEEE Trans. Automat. Control, AC-11 (1966), pp. 228–238, pp. 465–477.

[96] ———, *Optimal sensitivity and feedback; weighted seminorms, approximate inverses, and plant invariant schemes*, Proc. Allerton Conf., 1979.

[97] ———, *Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses*, IEEE Trans. Automat. Control, AC-23 (1981), pp. 301–320.

[98] G. ZAMES AND B. A. FRANCIS, *Feedback, minimax sensitivity, and optimal robustness*, IEEE Trans. Automat. Control, AC-28 (1983), pp. 585–601.

[99] G. ZAMES AND D. BENSOUSSAN, *Multivariable feedback, sensitivity, and decentralized control*, IEEE Trans. Automat. Control, AC-28 (1983), pp. 1030–1035.